

AN OBSERVATION OF THE SUBSPACES OF S'

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ABSTRACT. The spaces S'/\mathcal{P} equipped with the quotient topology and S'_∞ equipped with the weak-* topology are known to be homeomorphic, where \mathcal{P} denotes the set of all polynomials. The proof is a combination of the fact in the textbook by Treves and the well-known bipolar theorem. In this paper by extending slightly the idea employed in [5], we give an alternative proof of this fact and then we extend this proposition so that we can include some related function spaces.

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1. INTRODUCTION

It is useful to consider the quotient spaces of S' or \mathcal{D}' when we consider the homogeneous function spaces. Usually such a quotient space can be identified with some dual spaces as the following theorem shows:

Theorem 1.1. *Let X be a locally convex (Hausdorff) space whose topology is given by a family of semi-norms $\{p_\lambda\}_{\lambda \in \Lambda}$. Equip X^* with the weak-* topology. Let V be a closed subspace of X^* . Define the orthogonal space X_V to V by:*

$$X_V \equiv \bigcap_{x^* \in V} \ker(x^*)$$

and equip X_V with the topology induced by X . Then the topological dual X_V^ is isomorphic to X^*/V equipped with the quotient topology.*

The proof of Theorem 1.1 is a combination of [9, Propositions 35.5 and 35.6] and the bipolar theorem.

Theorem 1.2. [9, Propositions 35.5 and 35.6] *Let X be a locally convex Hausdorff space, and let N be a closed linear subspace of X . Then the kernel of the restriction X' to N' is*

$$N^\circ = \bigcap_{n \in N} \{x^* \in X' : \langle x', n \rangle = 0\}.$$

Furthermore, its quotient mapping is a homeomorphism from X'/N° to N' .

Theorem 1.3 (Bipolar theorem, [7, p. 126, Theorem]). *Let X be a Hausdorff topological vector space. Let V be a closed subspace of X^* equipped with the weak-* topology. Define*

$${}^\circ V \equiv \bigcap_{v^* \in V} \{x \in X : \langle v^*, x \rangle = 0\} = \bigcap_{v^* \in V} \ker(v^*).$$

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Then

$$V = \bigcap_{x \in {}^\circ V} \{v^* \in X^* : \langle v^*, x \rangle = 0\} (\equiv ({}^\circ V)^\circ).$$

If we let $V = \mathcal{P}(\subset \mathcal{S}')$, the linear space of all polynomials, then we can show that V is a closed subspace of \mathcal{S}' . One of the ways to check this is to use the Fourier transform. In fact, $f \in \mathcal{S}'$ belongs to \mathcal{P} if and only if the Fourier transform is supported in $\{0\}$.

The isomorphism $X^*/V \rightarrow X_V^*$ is given as follows: Let R be the natural restriction mapping $R : X^* \in f \mapsto f|_{X_V} \in X_V^*$. Denote by $\iota : X_V \rightarrow X$ the natural inclusion. Then ι and R are dual to each other and R is clearly continuous.

The aim of this paper is to give out an alternative proof of Theorem 1.1. We organize this short note as follows: We prove Theorem 1.1 in Section 2. After collecting some preliminary facts in Section 2.1 we plan to prove Theorem 1.1. We shall show that $\ker(R) = V$ in Section 2.2 which is essentially the bipolar theorem, that R is surjective in Section 2.3 and that R is an open mapping in Section 2.4.

We compare Theorem 1.1 with the existing results in Section 3. We devote Sections 3.1, 3.2, 3.3 and 3.4 to the application of Theorem 1.1 to the spaces \mathcal{S}'_∞ , \mathcal{S}'_m , \mathcal{S}'_e and \mathcal{D}' respectively. The definition of \mathcal{S}'_m and \mathcal{S}'_e will be given in Sections 3.2 and 3.3, respectively. For a topological space Y and its dual Y^* , we write $\langle y^*, y \rangle \equiv y^*(y)$ for the coupling of $y \in Y$ and $y^* \in Y^*$.

2. PROOF OF THEOREM 1.1

2.1. A reduction and preliminaries. Let X be a locally convex space whose topology is given by a family of semi-norms $\{p_\lambda\}_{\lambda \in \Lambda}$. Let us set

$$\mathcal{P} \equiv \left\{ \sum_{\lambda \in \Lambda_0} a_\lambda p_\lambda : \Lambda_0 \text{ is a finite subset of } \Lambda \text{ and } \{a_\lambda\}_{\lambda \in \Lambda_0} \subset \mathbb{N} \right\}.$$

Let $O \subset X^*$ be an open set. Then there exists $q \in \mathcal{P}$ such that

$$\{x \in X : q(x) < 1\} \subset O.$$

Therefore by replacing $\{p_\lambda\}_{\lambda \in \Lambda}$ with \mathcal{P} , we can assume that for any open set O there exists $\lambda(O) \in \Lambda$ such that

$$(2.1) \quad \{x \in X : p_{\lambda(O)}(x) < 1\} \subset O.$$

We invoke the propositions concerning the Hahn-Banach extension. First, we recall the following fact:

Proposition 2.1 (Geometric form, [7, p. 46]). *Let M be a linear subspace in a topological vector space L and let A be a non-empty convex, open subset of L , not intersecting M . Then there exists a closed hyperplane in L , containing M and not intersecting A .*

Next, we recall the Mazur theorem.

Proposition 2.2 (Analytic form, [14, p. 108, Theorem 3]). *Let X be a locally convex linear topological space, and M be a closed convex subset of X such that $a \cdot m \in M$ whenever $|a| \leq 1$ and $m \in M$. Then for any $x_0 \in X \setminus M$ there exists a continuous linear functional f on X such that $f(x_0) > 1 \geq |f(x)|$ for all $x \in M$.*

When M is a linear subspace in the above, $f(x) = 0$ for all $x \in M$. Thus, we can deduce the following well-known version:

Proposition 2.3 (Analytic form). *Let X be a topological vector space and let Y be a closed linear space. Then for any continuous linear functional ℓ_Y and $x \in X \setminus Y$ there exists a continuous linear functional ℓ_X such that $\ell_X|_Y = \ell_Y$ and that $\ell_X(x) = 0$.*

2.2. The kernel of R . We now specify $\ker(R)$. It is easy to see that $V \subset \ker(R)$ and that V and $\ker(R)$ are weak-* closed. Assume that V and $\ker(R)$ are different. Then by Proposition 2.1, we can find a continuous linear functional $\Phi^{**} : X^* \rightarrow \mathbb{C}$ such that $V \subset \ker(\Phi^{**})$ and

$$(2.2) \quad \ker(R) \cap \ker(\Phi^{**})^c \neq \emptyset.$$

Since $\Phi^{**} : X^* \rightarrow \mathbb{C}$ is continuous, we have

$$\begin{aligned} & \{x^* \in X^* : |\langle x^*, x_1 \rangle| < 1, |\langle x^*, x_2 \rangle| < 1, \dots, |\langle x^*, x_k \rangle| < 1\} \\ & \subset \{x^* \in X^* : |\langle \Phi^{**}, x^* \rangle| < 1\} \end{aligned}$$

for some $x_1, x_2, \dots, x_k \in X$. This means that

$$\bigcap_{j=1}^k \ker(Q_{x_j}) \subset \ker(\Phi^{**}),$$

where $Q : X \ni x \mapsto Q_x \in X^{**}$ is a natural inclusion. By the Helly theorem, we see that

$$\Phi^{**} = \sum_{j=1}^k a_j Q_{x_j} \in X^{**}.$$

Let $x^* \in V$ be arbitrary. Then we have

$$\left\langle x^*, \sum_{j=1}^k a_j x_j \right\rangle = \langle \Phi^{**}, x^* \rangle = 0,$$

since $x^* \in \ker(\Phi^{**})$. This means that

$$\sum_{j=1}^k a_j x_j \in X_V.$$

Let $x^* \in \ker(R)$. Then

$$\langle \Phi^{**}, x^* \rangle = \langle Q_{\sum_{j=1}^k a_j x_j}, x^* \rangle = \left\langle x^*, \sum_{j=1}^k a_j x_j \right\rangle = 0,$$

since $x^*|_{X_V} = 0$ and $\sum_{j=1}^k a_j x_j \in X_V$. Thus, $\ker(R) \subset \ker(\Phi^{**})$. This contradicts (2.2).

2.3. The surjectivity of $R : X \rightarrow X_V$. Let $z^* \in X_V^*$. Then $|\langle z^*, z \rangle| \leq p_\lambda(z)$ for some $\lambda \in \Lambda$ by (2.1). Use the Hahn-Banach theorem of analytic form to have a continuous linear functional x^* on X which extends z^* . Then $R(x^*) = z^*$.

2.4. The openness of R . We need the following lemma:

Lemma 2.4. *Let $z_1, z_2, \dots, z_k \in X_V$ and $x_1, x_2, \dots, x_l \in X$. Assume that the system $\{[x_1], [x_2], \dots, [x_l]\}$ is linearly independent in X/X_V . Then for all $z^* \in X_V$ satisfying $|\langle z^*, z_j \rangle| < 1$ for all $j = 1, 2, \dots, k$, we can find $x^* \in X_V$ so that z^* is a restriction of x^* and that $\langle x^*, x_j \rangle = 0$ for all $j = 1, 2, \dots, l$.*

Proof. We know that any linear space X has a norm $\|\cdot\|_X$ although it is not necessarily compatible with its original topological structure of X . For example, choose a Hamel basis $\{x_\theta\}_{\theta \in \Theta}$ and define

$$\left\| \sum_{\theta \in \Theta_0} a_\theta x_\theta \right\|_X \equiv \sum_{\theta \in \Theta_0} |a_\theta|$$

for any finite set Θ_0 .

Observe $\sum_{j=1}^l a_j x_j \notin X_V$ for any $(a_1, a_2, \dots, a_l) \neq (0, 0, \dots, 0)$, which yields $x_{(a_1, a_2, \dots, a_l)}^* \in X^*$ such that

$$(2.3) \quad \left\langle x_{(a_1, a_2, \dots, a_l)}^*, \sum_{j=1}^l a_j x_j \right\rangle = 1.$$

Since $x_{(a_1, a_2, \dots, a_l)}^* \in X^*$ is a continuous linear functional, we can find an index $\lambda(a_1, a_2, \dots, a_l) \in \Lambda$ such that

$$(2.4) \quad \{x \in X : p_{\lambda(a_1, a_2, \dots, a_l)}(x) < 1\} \subset \{x \in X : |\langle x_{(a_1, a_2, \dots, a_l)}^*, x \rangle| < 1\}.$$

Write

$$(2.5) \quad U_{(a_1, a_2, \dots, a_l)} \equiv \left\{ (b_1, b_2, \dots, b_l) \in \mathbb{C}^n : p_{\lambda(a_1, a_2, \dots, a_l)} \left(\sum_{j=1}^l (a_j - b_j) x_j \right) < 1 \right\}.$$

Since $S^{2l-1} \equiv \{(b_1, b_2, \dots, b_l) \in \mathbb{C}^l : |b_1|^2 + |b_2|^2 + \dots + |b_l|^2 = 1\}$ is a compact set, we can find a finite covering $\{U_{(a_1, a_2, \dots, a_l)}\}_{(a_1, a_2, \dots, a_l) \in A}$ of S^{2l-1} , where A is a finite subset of S^{2l-1} .

Let us denote by X_A and $X_{A,V}$ the completion of X and X_V with respect to the norm

$$\|\cdot\|_A \equiv \|\cdot\|_X + \sum_{(a_1, a_2, \dots, a_l) \in A} p_{\lambda(a_1, a_2, \dots, a_l)}(\cdot),$$

respectively. Then we can extend $x_{(a_1, a_2, \dots, a_l)}^*$ to a continuous linear functional $\mathfrak{X}_{(a_1, a_2, \dots, a_l)}^*$ to $X_{A,V}$.

Note that for any $(b_1, b_2, \dots, b_l) \in \mathbb{C}^n \setminus (0, 0, \dots, 0)$, there exists $(a_1, a_2, \dots, a_l) \in A$ such that

$$\frac{1}{\sqrt{|b_1|^2 + |b_2|^2 + \dots + |b_l|^2}} (b_1, b_2, \dots, b_l) \in U_{(a_1, a_2, \dots, a_l)}.$$

For such an element $(a_1, a_2, \dots, a_l) \in A$, we deduce

$$x_{\lambda(a_1, a_2, \dots, a_l)}^* \left(\sum_{j=1}^k a_j x_j \right) \neq 0$$

from (2.3)–(2.5). Since $\mathfrak{X}_{\lambda(a_1, a_2, \dots, a_l)}^*(x) = 0$ for any $x \in X_{V,A}$, we see that $\{[x_1], [x_2], \dots, [x_l]\}$ is a linearly independent system in $X_A/X_{A,V}$. Then use the

Hahn-Banach theorem of geometric form (see Proposition 2.3). Then we obtain a continuous functional $\mathfrak{X}^* : X_A \rightarrow \mathbb{C}$ so that $\mathfrak{X}^*(x_j) = 0$ for $j = 1, 2, \dots, l$. If we set $\mathfrak{X}^*|_X = x^*$, then we have the desired result. \square

The following proposition shows that R is an open mapping:

Proposition 2.5. *Let $z_1, z_2, \dots, z_k \in X_V$ and $x_1, x_2, \dots, x_L \in X$. Then there exists $r > 1$ depending only on $z_1, z_2, \dots, z_k \in X_V$ and $x_1, x_2, \dots, x_L \in X$ such that for all $z^* \in X_V$ satisfying $|\langle z^*, z_j \rangle| < 1$ for all $j = 1, 2, \dots, k$, we can find $x^* \in X_V$ so that z^* is a restriction of x^* and that $|\langle x^*, x_j \rangle| < r$ for all $j = 1, 2, \dots, L$. Namely, the range*

$$\bigcap_{j=1}^k \{x^* \in X^* : |\langle x^*, z_j \rangle| < 1\} \cap \bigcap_{j=1}^l \{x^* \in X^* : |\langle x^*, x_j \rangle| < r\}$$

by R contains

$$\bigcap_{j=1}^k \{x^* \in X^* : |\langle x^*, z_j \rangle| < 1\}.$$

In particular, the range

$$\bigcap_{j=1}^k \{x^* \in X^* : |\langle x^*, z_j \rangle| < 1\} \cap \bigcap_{j=1}^l \{x^* \in X^* : |\langle x^*, x_j \rangle| < 1\}$$

by R contains

$$\bigcap_{j=1}^k \{x^* \in X^* : |\langle x^*, z_j \rangle| < r^{-1}\}.$$

Proof. Let us assume that $\{[x_1], [x_2], \dots, [x_l]\}$ is a maximal linearly independent family in X/X_V , where $l \leq L$. Then for any $j \in (l, L] \cap \mathbb{N}$ and $k \in [1, l] \cap \mathbb{N}$, we can find $\mu_{jk} \in \mathbb{C}$ and $\tilde{z}_j \in X_V$ such that

$$x_j = \tilde{z}_j + \sum_{m=1}^l \mu_{jm} x_m.$$

Let

$$r \equiv 1 + \max_{j=1,2,\dots,l} |\langle z^*, \tilde{z}_j \rangle|$$

Note that

$$|\langle z^*, z_j \rangle|, |\langle z^*, \tilde{z}_m \rangle| < r$$

for all $j = 1, 2, \dots, k$ and $m = l+1, l+2, \dots, L$. According to Lemma 2.4, we can find $x^* \in X^*$ so that x^* is an extension of z^* and that

$$\langle z^*, x_j \rangle = 0, \quad j \in (l, L].$$

Thus, we obtain the desired result. \square

3. APPLICATIONS

3.1. Schwartz space. The Schwartz space \mathcal{S} is defined to be the set of all $\Phi \in C^\infty$ for which the semi-norm $p_N(\Phi)$ is finite for all $N \in \mathbb{N}_0 \equiv \{0, 1, \dots\}$, where

$$p_N(\Phi) \equiv \sum_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha \Phi(x)|.$$

The space \mathcal{S}_∞ is the set of all $\Phi \in \mathcal{S}$ for which

$$\int_{\mathbb{R}^n} x^\alpha \Phi(x) dx = 0$$

for all $\alpha \in \mathbb{N}_0^n$.

The topological dual of \mathcal{S} and \mathcal{S}_∞ are denoted by \mathcal{S}' and \mathcal{S}'_∞ , respectively. The elements in \mathcal{S}' and \mathcal{S}'_∞ are called Schwarz distributions and Lizorkin distributions, respectively. Equip \mathcal{S}' and \mathcal{S}'_∞ with the weak-* topology.

Since \mathcal{S}_∞ is continuously embedded into \mathcal{S} , the dual operator R , called the restriction, is continuous from \mathcal{S}' to \mathcal{S}'_∞ . We can generalize the following fact and refine the proof:

Theorem 3.1. [5, Theorem 6.18] *The restriction mapping $R : F \in \mathcal{S}' \mapsto F|_{\mathcal{S}'_\infty} \in \mathcal{S}'_\infty$ is open, namely the image $R(U)$ is open in \mathcal{S}'_∞ for any open set U in \mathcal{S}' .*

The statement can be found in [8], where Triebel applied this theorem to the definition of homogeneous function spaces. Note also that Holschneider considered Theorem 3.1 in the context of wavelet analysis in [3, Theorem 24.0.4], where he applied a general result [9, Propositions 35.5 and 35.6] to this special setting. We can find the proof of Theorem 3.1 in [15, Proposition 8.1]. But there is a gap in Step 4, where the openness of R is proved using the closed graph theorem. It seems that the closed graph theorem is not applicable to the space \mathcal{S}' . Our proof reinforces Step 4 in the proof of [15, Proposition 8.1].

According to the proof of Theorem 1.1, there is no need to use the Fourier transform.

3.2. The space \mathcal{S}'_m . We recall the definition of $\mathcal{S}'/\mathcal{P}_m$, where \mathcal{P}_m denotes the set of all polynomials of degree less than or equal to m . Following Bourdaud [1], we denote by \mathcal{S}_m the orthogonal space of \mathcal{P}_m in \mathcal{S} and by \mathcal{S}'_m its topological dual. See [4, 12, 13, 16] for applications to homogeneous function spaces defined in [10, 12, 15].

3.3. The Hasumi space \mathcal{S}'_e . Let $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$. Write temporarily $\varphi_{(N;\alpha)}(x) \equiv e^{N|x|} \partial^\alpha \varphi(x)$ ($x \in \mathbb{R}^n$) for $\varphi \in C^\infty$. Define \mathcal{S}_e as follows:

$$\mathcal{S}_e \equiv \bigcap_{N \in \mathbb{N}, \alpha \in \mathbb{N}_0^n} \{ \varphi \in C^\infty : \varphi_{(N;\alpha)} \in L^\infty \}.$$

The topological dual is denoted by \mathcal{S}'_e is called the Hasumi space [2]. An analogy to the spaces \mathcal{S}' and \mathcal{S}'_m is available. We refer to [6] for function spaces contained in \mathcal{S}'_e .

3.4. The space \mathcal{D}' . A similar thing to Sections 3.1–3.3 applies to \mathcal{D}' . If we define

$$\mathcal{D}_m \equiv \left\{ \varphi \in \mathcal{D} : \int_{\mathbb{R}^n} x^\alpha \varphi(x) dx = 0 \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq m \right\},$$

then we have

$$\mathcal{D}'_m \sim \mathcal{D}'/\mathcal{P}_m.$$

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